An arithmetic-geometric correspondence for character stacks

(based on arXiv: 2309.15331, joint work with M. Hablicsek and Á. González-Prieto)

Γ = finitely generated group

G = (linear) algebraic group (e.g. GL_n, SL_n)

<u>Def</u> G-representation variety $R_G(\Gamma) = \mathrm{Hom}(\Gamma, G)$ (relations in Γ yield algebraic equations)

<u>Ex</u> $\Gamma = \pi_1(M)$ for M a compact connected manifold

- $\Gamma = \pi_1(M) = \mathbb{Z}$, $\underbrace{R_G(S^1)}_{\text{shorthand notation}} = G$
- $\Gamma = \pi_1(\Sigma_g) = \langle a_1, b_1, \dots, a_g, b_g \mid \prod_{i=1}^{g} [a_i, b_i] = 1 \rangle$ (main example)

$$R_G(\Sigma_g) = \left\{ (A_1, B_1, \dots, A_g, B_g) \in G^{2g} \;\middle|\; \prod_{i=1}^{g} [A_i, B_i] = 1 \right\} \subseteq G^{2g}$$

(closed subvariety)

Why $\Gamma = \pi_1(M)$?

$$\mathrm{Hom}(\pi_1(M), G) \longleftrightarrow \{ G\text{-local systems on } M \}$$

conjugate representations $\longleftrightarrow$ isomorphic local systems

<u>Def</u> G-character stack $\mathfrak{X}_G(M) := [R_G(M) / G]$ (stacky quotient) (G acts by conjugation)

<u>This talk</u>

1) Introduce two methods for computing algebraic / cohomological invariants of $\mathfrak{X}_G(\Sigma_g)$
(arithmetic & geometric method)

2) Show there is a common framework

Arithmetic method (Hausel, Rodriguez-Villegas)

Idea Count $\mathbb{F}_q$-points $\#R_G(\Gamma)(\mathbb{F}_q) = \#R_{G(\mathbb{F}_q)}(\Gamma)$

Frobenius' formula: If G is a finite group, then

$$\#R_G(\Sigma_g) = \#G \cdot \sum_{\chi \in \hat{G}} \left(\frac{\#G}{\chi(1)}\right)^{2g-2}$$

Ex $(g=1)$

$$\#\{(A,B)\in G^2 \mid [A,B]=1\} = \sum_{A\in G} \#\,\mathrm{Cent}(A)$$
$$= \sum_{A\in G} \frac{\#G}{\#\,\mathrm{Conj}(A)} \qquad \text{(orbit-stabilizer)}$$
$$= \#G \cdot \#\text{ conj. classes of } G$$
$$= \#G \cdot \#\hat{G}$$

Theorem (Katz) If X is a complex variety, and $\#X(\mathbb{F}_q)$ is polynomial in q, — this condition will always hold for us

then this polynomial is the $\underbrace{\text{E-polynomial of } X}$, with $q = uv$.

actually, one needs to take a model for X over a fin. gen. $\mathbb{Z}$-algebra

$$\mathbb{Z}[u,v] \ni e(X) = \sum_{k,p,q} (-1)^k \underbrace{h_c^{k;p,q}(X)}_{\substack{\text{mixed Hodge} \\ \text{numbers of } X}} u^p v^q$$

$$\overset{\substack{\text{if } X \\ \text{sm. proj.}}}{=} \sum_{p,q} (-1)^{p+q} \dim_{\mathbb{C}} H^q(X, \Omega_X^p)\, u^p v^q$$

Ex $e(\mathbb{A}^n) = (uv)^n$

$e(\mathbb{P}^n) = 1 + uv + \cdots + (uv)^n$ (agrees with the Hodge diamond)

Note Usually, $e(\mathcal{X}_G(M)) = e(R_G(M)) / e(G)$

Geometric method (Gonzalez-Prieto, Logares, Muñoz, Newstead)

Idea Compute E-polynomial of $R_G(\Sigma_g)$ using following properties:

- (cut-and-paste) $e(X) = e(Z) + e(X \setminus Z)$ for closed subvarieties $Z \subseteq X$
- (multiplicative) $e(X \times Y) = e(X)\, e(Y)$

note: compatible with Katz' theorem

Might as well compute invariant in the Grothendieck ring of varieties:

$$K_0(\mathrm{Var}_k) = \bigoplus_{\substack{\text{isom. classes} \\ [X] \text{ of varieties}/k}} \mathbb{Z} \Big/ [X] = [Z] + [X/Z]$$

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Method tries to make smart stratifications to understand

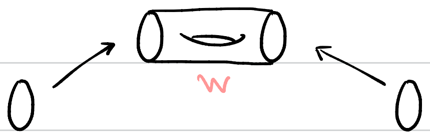
$$\left[\begin{array}{c} G^2 \longrightarrow G \\ (A,B) \longmapsto [A,B] \end{array}\right] \in K_0(\mathrm{Var}/G)$$

(Will not go into detail, but there is a way to glue these classes together to get a product of commutators)

Topological Quantum Field Theories

Let's cut Σ_g into pieces:

$\Sigma_g =$ 



$$\begin{tikzcd} & \mathfrak{X}_G(W) \arrow[dl, "\text{restriction}"'] \arrow[dr, "\text{restriction}"] & \\ \mathfrak{X}_G(S^1) & & \mathfrak{X}_G(S^1) \end{tikzcd}$$

(remark on composition)

$$\overset{n}{\cancel{2}}\text{-Bord} \xrightarrow[\text{(field theory)}]{\mathcal{F}} \mathrm{Corr}(\mathrm{Stck})$$

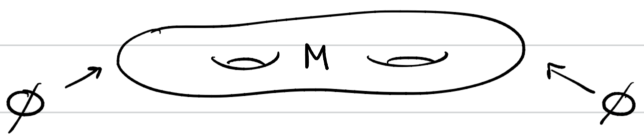
$$\begin{tikzcd} & T \times_X Z \arrow[dl] \arrow[d] \arrow[ddrr, bend left, "g \circ \pi_Z"] & & \\ T \arrow[d] & Z \arrow[dl, "f"'] \arrow[dr, "g"] & & \\ X & & Y & \end{tikzcd}$$

$$K_0(\mathrm{Stck}/X) \xrightarrow{f^*} K_0(\mathrm{Stck}/Z) \xrightarrow{g_!} K_0(\mathrm{Stck}/Y)$$

$$\mathrm{Corr}(\mathrm{Stck}) \xrightarrow[\text{(quantization)}]{\mathcal{Q}} K_0(\mathrm{Stck})\text{-Mod}$$

Def The composition $Z = \mathcal{Q} \circ \mathcal{F}$ is a (lax) monoidal functor $n\text{-Bord} \to K_0(\mathrm{Stck})\text{-Mod}$ that is, a Topological Quantum Field Theory (TQFT)

Note



$\overset{Z}{\longmapsto}$

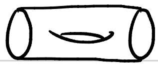
$$K_0(\mathrm{Stck}/\cancel{\mathfrak{X}_G(\emptyset)}_{\,k}) \to K_0(\mathrm{Stck}/\mathfrak{X}_G(M)) \to K_0(\mathrm{Stck}/k)$$

$$1 = [k] \longmapsto [\mathfrak{X}_G(M) \xrightarrow{id} \mathfrak{X}_G(M)] \longmapsto [\mathfrak{X}_G(M)]$$

We say Z quantizes $[\mathfrak{X}_G(M)]$

Arithmetic TQFT

Let's repeat the construction, but with G a finite group

$\emptyset \rightarrow$  $\leftarrow \emptyset$

$$\mathscr{X}_G(-) \leftarrow \mathscr{X}_G(-) \rightarrow \mathscr{X}_G(-)$$

$$n\text{-Bord} \xrightarrow[\text{(arithmetic field theory)}]{\mathcal{F}^\#} \mathrm{Corr}(\mathrm{Grpd})$$

$$A \leftarrow C \rightarrow B \qquad\qquad \mathbb{C}^A \xrightarrow{\text{restriction}} \mathbb{C}^C \xrightarrow{\text{integration along fibers}} \mathbb{C}^B$$

$$\mathrm{Corr}(\mathrm{Grpd}) \xrightarrow[\text{(arithmetic quantization)}]{Q^\#} \mathbb{C}\text{-Vect}$$

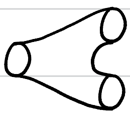
Result The arithmetic TQFT $Z^\# = Q^\# \circ \mathcal{F}^\#$

quantizes $\#\mathscr{X}_G(M) = \#R_G(M) / \#G$ (groupoid cardinality)

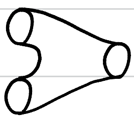
How is this related to the arithmetic method ?

Ex $(n=2)$ $\quad S^1 \overset{\mathcal{F}^\#}{\longmapsto} \mathscr{X}_G(S^1) = \underset{\text{(conjugation)}}{G/G} \overset{Q^\#}{\longmapsto} \mathbb{C}^{G/G} \cong R_{\mathbb{C}}(G)$

representation ring of G

 $\mapsto \mu$ = convolution

 $\mapsto \eta$ = unit

 $\mapsto \delta$ = comultiplication

 $\mapsto \varepsilon$ = counit

$\Longrightarrow$ Frobenius algebra = algebra + coalgebra + compatibilities

Working out $\mu, \eta, \delta, \varepsilon$ in terms of characters, we find

$$Z^\#(\Sigma_g) = \varepsilon \circ (\mu \circ \delta)^g \circ \eta = \sum_{\chi \in \hat{G}} \left(\frac{\#G}{\chi(1)}\right)^{2g-2} = \frac{\#R_G(\Sigma_g)}{\#G}$$

Note The arithmetic TQFT $Z^\#$ generalizes the arithmetic method in the sense that it works in any dimension

Arithmetic-geometric correspondence

$$\begin{tikzcd}[column sep=large]
\text{Geometric:} & n\text{-Bord} \arrow[r, "\mathcal{F}"] \arrow[d, equal] \arrow[dr, phantom, "\Downarrow\wr\ \text{if } G \text{ is connected}"] & \mathrm{Corr}(\mathrm{Stck}) \arrow[r, "Q"] \arrow[d, "(-)(\mathbb{F}_q)"] \arrow[dr, phantom, "\Downarrow"] & K_0(\mathrm{Stck})\text{-}\mathrm{Mod} \\
\text{Arithmetic:} & n\text{-Bord} \arrow[r, "\mathcal{F}^\#"] & \mathrm{Corr}(\mathrm{Grpd}) \arrow[r, "Q^\#"] & \mathbb{C}\text{-}\mathrm{Vect} \arrow[u, "\text{restriction of scalars}"']
\end{tikzcd}$$

Counting $\mathbb{F}_q$-points :
$$\begin{aligned} K_0(\mathrm{Stck}) &\longrightarrow \mathbb{C} \\ [X] &\longmapsto \# X(\mathbb{F}_q) \end{aligned} \qquad (\text{ring morphism})$$

RHS Counting points in fibers:
$$\begin{aligned} K_0(\mathrm{Stck}/X) &\longrightarrow \mathbb{C}^{X(\mathbb{F}_q)} \\ [Y \xrightarrow{f} X] &\longmapsto (x \mapsto \# f^{-1}(x)) \end{aligned}$$

Prop This defines a natural transformation "$\Downarrow$"

LHS $[R_G(M)/G](\mathbb{F}_q)$ vs. $[R_{G(\mathbb{F}_q)}(M) \,/\, G(\mathbb{F}_q)]$

Prop If G is connected, then these groupoids are (naturally) equivalent (Lang's theorem)

Theorem If G is connected, then there is a natural transformation $Z \Rightarrow Z^\#$

Corollaries

Geometric $\Longrightarrow$ Arithmetic

Under the natural transformation,

(1) the eigenvalues of $Z($$)$ are sent to $\#G(\mathbb{F}_q)/\chi(1)$ for the irreducible characters χ of $G(\mathbb{F}_q)$,

(2) the eigenvectors of $Z($$)$ are sent to the sums of equi-dimensional charactes of $G(\mathbb{F}_q)$, that is, $\sum_{\substack{\chi \in \hat{G} \\ \text{s.t. } \chi(1)=d}} \chi$

Note From geometric computations, we can deduce (partially) information on the character table of $G(\mathbb{F}_q)$

Arithmetic $\Longrightarrow$ Geometric

No formal implication, but it seems one can lift arithmetic eigenvalues / eigenvectors to geometric ones.

Q : Are eigenvalues of $Z($$)$ always polynomial in $[\mathbb{A}^1]$?

Q : Is $[R_G(\Sigma_g)]$ always polynomial in $[\mathbb{A}^1]$?